NOTES FROM THE WORKSHOP ON APPROXIMATE GROUPS AND APERIODIC ORDER

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Based on joint work with Michael Björklund

1. Heat kernel hyperuniformity

Let Λ be an invariant (locally square-summable) point process on hyperbolic space

$$\mathbb{H}^n = \mathrm{SO}^+(1,n)/\mathrm{SO}(n)$$

and let σ_Λ denote the associated (centered) diffraction measure on $(0,+\infty) \cup i[0,\rho) \subset \mathbb{C}$, where $\rho = (n-1)/2$. Recall from the talk that σ_Λ is the unique positive Radon measure on $(0,+\infty) \cup i[0,\rho)$ satisfying

$$\mathrm{Var}\Big(\sum_{p\in\Lambda} f(p)\Big) = \int_0^\infty |\widehat{f}(\lambda)|^2 d\sigma_\Lambda^{(p)}(\lambda) + \int_0^\rho |\widehat{f}(i\lambda)|^2 d\sigma_\Lambda^{(c)}(\lambda)$$

for all compactly supported continuous functions f on $\mathbb{H}^n$, where $\sigma_\Lambda^{(p)}, \sigma_\Lambda^{(c)}$ are the restrictions of σ_Λ to $(0,+\infty)$ and $i[0,\rho)$ respectively.

Assumption: We will assume throughout this note that $\sigma_\Lambda^{(c)} = 0$, saving us from any "super-Poissonian" density fluctuations of the process.

Goal: We want show that such a point process Λ is spectrally hyperuniform, i.e.

$$\sigma_\Lambda([0,\varepsilon]) = o(\varepsilon^3)\,,$$

if and only if

$$\limsup_{t\to+\infty} t^{3/2} \mathrm{e}^{2\rho^2 t}\, \mathrm{Var}\Big(\sum_{p\in\Lambda} h_t(p)\Big) = 0\,, \tag{1.1}$$

where

$$h_t(x) = \int_0^\infty \mathrm{e}^{-t(\rho^2+\lambda^2)} \phi_\lambda(d(x,o)) \frac{d\lambda}{|c_n(\lambda)|^2}$$

is the heat kernel on $\mathbb{H}^n$. If Λ satisfies equation (1.1), then we say that it is *heat kernel hyperuniform*.

1.1. Spectral hyperuniformity implies heat kernel hyperuniformity

Assume that Λ is spectrally hyperuniform. We will first rewrite the variance of the periodized heat kernel into a suitable form and then formulate Lemma 1.1, where the heart of the proof lies.

Consider the function $f_t = \mathrm{e}^{\rho^2 t} h_t$ on $\mathbb{H}^n$, so that

$$\widehat{f_t}(\lambda) = \mathrm{e}^{\rho^2 t}\widehat{h}_t(\lambda) = \mathrm{e}^{\rho^2 t}\mathrm{e}^{-t(\rho^2+\lambda^2)} = \mathrm{e}^{-t\lambda^2} =: \psi(\sqrt{t}\lambda)\,.$$

Using the diffraction measure σ_Λ, we rewrite the heat kernel variance as

$$t^{3/2}\mathrm{e}^{2\rho^2 t}\, \mathrm{Var}\Big(\sum_{p\in\Lambda} h_t(p)\Big) = t^{3/2}\, \mathrm{Var}\Big(\sum_{p\in\Lambda} f_t(p)\Big) = t^{3/2}\int_0^\infty \psi(\sqrt{t}\lambda)^2 d\sigma_\Lambda(\lambda)\,.$$

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Letting $s = \sqrt{t}$, we have to show that

$$s^3 \int_0^\infty \psi(s\lambda)^2 d\sigma_\Lambda(\lambda) \longrightarrow 0\ , \quad s \to +\infty\,.$$

Set

$$\varphi(r) = -2\psi'(r)\psi(r) = 4r\,\mathrm{e}^{-2r^2}$$

so that

$$\psi(s\lambda)^2 = s\int_\lambda^\infty \varphi(sr)dr\,.$$

By a standard change of variables,

$$\int_0^\infty \psi(s\lambda)^2 d\sigma_\Lambda(\lambda) = s\int_0^\infty \Big(\int_\lambda^\infty \varphi(sr)dr\Big)d\sigma_\Lambda(\lambda) = s\int_0^\infty \varphi(sr)\Big(\int_0^t d\sigma_\Lambda(\lambda)\Big)dr$$
$$= s\int_0^\infty \varphi(sr)\sigma_\Lambda([0,r])dr\,.$$

This means that it is enough to show that

$$s^4\int_0^\infty \varphi(sr)\sigma_\Lambda([0,r])dr \longrightarrow 0\,, \quad s \to +\infty\,.$$

The core of the proof lies in the following Lemma.

Lemma 1.1 (Björklund). *Suppose that there are constants $\alpha,\beta > 0$, $\gamma \in (0,1)$, $A, B, C \geq 0$ and Borel functions $\omega, \varphi : [0,+\infty) \to [0,\infty)$ satisfying*

(i) ρ is increasing and such that

- *for every $\varepsilon > 0$ there is an $r_\varepsilon \in (0,1)$ with $r_\varepsilon \to 0$ such that $\omega(r) \leq A\varepsilon r^\alpha$ for all $r \in [0, r_\varepsilon^{1-\gamma}]$,*
- *there is a constant $M > 0$ such that $\omega(r) \leq Br^\beta$ for all $r \geq M$.*

(ii) φ is such that

(1) $\int_0^\infty r^\alpha \varphi(r)dr < +\infty$,

(2) $R^{\alpha/\gamma}\int_R^\infty \varphi(r)dr \to 0$ *as* $R \to +\infty$,

(3) $R^{\alpha-\beta}\int_R^\infty r^\beta\varphi(r)dr \to 0$ *as* $R \to +\infty$.

Then

$$r_\varepsilon^{-(\alpha+1)}\int_0^\infty \varphi(r/r_\varepsilon)\omega(r)dr \longrightarrow 0\,, \quad \varepsilon \to 0^+\,.$$

With this Lemma in mind, we set $\varphi(r) = 4r\mathrm{e}^{-2r^2}$ and $\omega(r) = \sigma_\Lambda([0,r])$. By the assumed spectral hyperuniformity, there is a constant $A \geq 0$ and a sequence $r_\varepsilon \to 0$ such that

$$\omega(r) \leq A\varepsilon r^3\,, \quad r \in [0, r_\varepsilon^{1/2}]\,,$$

so we can take $\alpha = 3$, $\gamma = 1/2$. Finding constants B, β as in the Lemma above requires some more work.

Lemma 1.2. *Let Λ be an invariant point process on $\mathbb{H}^n$ with diffraction σ_Λ. Then $\sigma_\Lambda([0,R]) \ll R^n$ for sufficiently large $R > 0$.*

Remark 1.3. When $\omega(r) = \sigma([0,r])$ is the diffraction measure of a point process, Lemma 1.1 is applicable for all rank one spaces. In particular, the crucial constants for Euclidean and hyperbolic space are

G/K	$\mathbb{R}^n$	$\mathbb{H}^n$
α	n	3
β	n	n

Thus for hyperbolic space there is a $B > 0$ and an $M > 0$ such that $\rho(r) \leq Br^n$ for all $r \geq M$, so take $\beta = n$.

Modulo the proof of Lemma 1.1, it suffices to verify points (1)-(3) for the function $\varphi(r) = 4r\mathrm{e}^{-2r^2}$.

(1) we have

$$4\int_0^\infty r^4 \mathrm{e}^{-2r^2} dr < +\infty\,.$$

(2)

$$R^6 \int_R^\infty 4r\mathrm{e}^{-2r^2} dr = 2R^6 \mathrm{e}^{-2R^2} \longrightarrow 0\,, \quad R \to +\infty\,.$$

(3)

$$R^{3-n} \int_R^\infty r^n \mathrm{e}^{-2r^2} dr = R^4 \int_1^\infty r^n \mathrm{e}^{-2R^2 r^2} dr \longrightarrow 0\,, \quad R \to +\infty$$

by dominated convergence.

Letting $s_\varepsilon = r_\varepsilon^{-1}$, then Lemma 1.1 tells us that

$$s_\varepsilon^4 \int_0^\infty \varphi(s_\varepsilon r)\sigma_\Lambda([0,r])dr \longrightarrow 0\,, \quad \varepsilon \to 0^+\,,$$

which is what we wanted to show.

Proof of Lemma 1.1. Let $I_\varepsilon = \int_0^\infty \varphi(r/r_\varepsilon)\rho(r)dr$. Then we use the assumed monotone growth and estimates on ω to find that

$$I_\varepsilon \leq \int_0^{r_\varepsilon^{1-\gamma}} \varphi(r/r_\varepsilon)A\varepsilon r^\alpha dr + \omega(M)\int_{r_\varepsilon^{1-\gamma}}^M \varphi(r/r_\varepsilon)dr + \int_M^\infty \varphi(r/r_\varepsilon)Br^\beta dr =: I_1 + I_2 + I_3\,.$$

We need to show that $I_j = o(r_\varepsilon^{\alpha+1})$ for $j = 1,2,3$ as $\varepsilon \to 0^+$.

$$I_1 \leq A\varepsilon r_\varepsilon^{\alpha+1} \int_0^\infty \varphi(r) r^\alpha dr$$

$$I_2 \leq \varphi(M) r_\varepsilon \int_{r_\varepsilon^{-\gamma}}^\infty \varphi(r)dr$$

$$I_3 = Br_\varepsilon^{\beta+1} \int_{M/r_\varepsilon}^\infty \varphi(r)r^\beta dr\,.$$

Dividing by $r_\varepsilon^{\alpha+1}$, we get that

$$\frac{I_\varepsilon}{r_\varepsilon^{\alpha+1}} = \frac{I_1+I_2+I_3}{r_\varepsilon^{\alpha+1}} \ll \varepsilon + \omega(M)\underbrace{(r_\varepsilon^{-\gamma})^{\alpha/\gamma}\int_{r_\varepsilon^{-\gamma}}^\infty \varphi(r)dr}_{\longrightarrow 0} + \underbrace{r_\varepsilon^{-(\alpha-\beta)}\int_{M/r_\varepsilon}^\infty \varphi(r)r^\beta dr}_{\longrightarrow 0}\,.$$

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1.2. **Heat kernel hyperuniformity implies spectral hyperuniformity**

Assume that equation (1.1) holds. We will show that $\sigma_\Lambda([0,\varepsilon]) = o(\varepsilon^3)$ as $\varepsilon \to 0^+$.

This is fortunately quite straightforward: Let $\delta_t > 0$ and bound canonically from above,

$$\mathrm{Var}\Big(\sum_{p\in\Lambda} h_t(p)\Big) = \int_0^\infty |\widehat{h}_t(\lambda)|^2 d\sigma_\Lambda(\lambda) \geq \int_0^{\delta_t} |\widehat{h}_t(\lambda)|^2 d\sigma_\Lambda(\lambda) = \mathrm{e}^{-2\rho^2 t}\int_0^{\delta_t} \mathrm{e}^{-2t\lambda^2} d\sigma_\Lambda(\lambda)\,.$$

Assuming heat kernel hyperuniformity, there is for every $\varepsilon > 0$ a $t_\varepsilon > 0$ such that

$$t^{3/2} \int_0^{\delta_t} \mathrm{e}^{-2t\lambda^2} d\sigma_\Lambda(\lambda) < \varepsilon\,, \quad \forall\, t \geq t_\varepsilon\,.$$

In particular,

$$t^{3/2} \mathrm{e}^{-2t\delta_t^2} \sigma_\Lambda([0,\delta_t]) < \varepsilon\,, \quad \forall\, t \geq t_\varepsilon\,,$$

so the question boils down to the following: If we can find $\delta_t \to 0$ as $t \to +\infty$ such that

$$\delta_t^{-3} \ll t^{3/2} \mathrm{e}^{-2t\delta_t^2}\,, \quad \forall\, t \geq t_\varepsilon\,, \tag{1.2}$$

then

$$\frac{\sigma_\Lambda([0,\delta_t])}{\delta_t^3} \ll t^{3/2} \mathrm{e}^{-2t\delta_t^2} \sigma_\Lambda([0,\delta_t]) < \varepsilon\,, \quad \forall\, t \geq t_\varepsilon\,,$$

as desired.

Finding a solution δ_t to equation (1.2) is equivalent to $2t\delta_t^2 - 3\log(\delta_t) \leq \frac{3c}{2}\log(t)$ for some $c > 0$ for all sufficiently large t. As an example, by setting $c = 2$ we get $2t\delta_t^2 \leq 3\log(t\delta_t)$, for which $\delta_t = t^{-1/2}$ is an example of a solution whenever $t \geq \mathrm{e}^{4/3}$.